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The Multi-variable Arakawa–Kaneko Zeta Function for Non-positive Indices and Its Values at Non-positive Integers

by

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Abstract. The multi-variable Arakawa–Kaneko zeta function is defined in a suitable way. This function can be regarded as the one to be paired up with the multi-variable η-function defined by Kaneko and Tsumura. It is shown that the multi-variable Arakawa–Kaneko zeta function is analytically continued to an entire function, and its values at non-positive integers satisfy a certain duality formula which is a generalization of that for the poly-Bernoulli numbers of C-type.

1. Introduction

For an integer $k \in \mathbb{Z}$, the poly-Bernoulli numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}$$

(See [4] and [1]). Here, $\mathrm{Li}_k(z)$ is the polylogarithm function defined by

$$\mathrm{Li}_k(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

for $k \in \mathbb{Z}$. In the case $k = 1$, the numbers $\{B_n^{(1)}\}$ and $\{C_n^{(1)}\}$ are coincide with the classical Bernoulli numbers which are usually written by $\{B_n\}$ and $\{C_n\}$, respectively. The numbers $\{B_n\}$ and $\{C_n\}$ are almost the same. Indeed, it holds that

$$B_n = (-1)^n C_n \quad (n \in \mathbb{Z}_{\geq 0}).$$

In particular,

$$B_1 = \frac{1}{2} = -C_1, \quad B_n = C_n = 0 \ (n \geq 3 : \text{odd}).$$

In general, however, there is considerable difference between $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$. Therefore, we distinguish and call them *B-type* and *C-type*, respectively.

Various properties, such as a recurrence relation, an expression of generating series or an explicit formula in terms of the Stirling numbers of the second kind, have been studied. One of the basic results is the duality formula ([4, Theorem 2], [5, Section 2]), which states for $k, m \in \mathbb{Z}_{\geq 0}$,

$$B_m^{(-k)} = B_k^{(-m)}\,, \tag{1}$$

$$C_m^{(-k-1)} = C_k^{(-m-1)}\,. \tag{2}$$

In 1999, Arakawa and Kaneko defined the following zeta function:

$$\xi(k; s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} dt \quad (\Re(s) > 0)$$

for $k \in \mathbb{Z}_{\geq 1}$, in the context of finding a zeta function whose values are expressed by the poly-Bernoulli numbers. Indeed, it is shown that $\xi(k; s)$ is analytically continued to an entire function, and values at non-positive integers are written by the poly-Bernoulli numbers of C-type ([1, Theorem 6 (i)]):

$$\xi(k; -n) = (-1)^n C_n^{(k)} \quad (n \in \mathbb{Z}_{\geq 0})\,. \tag{3}$$

Naturally, we have a question of whether the poly-Bernoulli numbers *of B-type* appear as values of zeta functions of any sort. Kaneko and Tsumura answered this question in a more general setting, by introducing the following zeta function:

$$\eta(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{t})}{1 - e^t} dt \quad (\Re(s) > 1-r) \tag{4}$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$. Here, $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ is the multiple polylogarithm function defined by

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) := \sum_{0<m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}$$

for $k_1, \ldots, k_r \in \mathbb{Z}$. Indeed, it is shown that $\eta(k_1, \ldots, k_r; s)$ is analytically continued to an entire function, and values at non-positive integers are written by the multi-poly-Bernoulli numbers of B-type ([6, Theorem 2.3]):

$$\eta(k_1, \ldots, k_r; -n) = B_n^{(k_1,\ldots,k_r)} \quad (n \in \mathbb{Z}_{\geq 0})\,. \tag{5}$$

Here, the multi-poly-Bernoulli numbers $\{B_n^{(k_1,\ldots,k_r)}\}$ and $\{C_n^{(k_1,\ldots,k_r)}\}$ are further generalizations of the poly-Bernoulli numbers, which are defined by

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}\,,$$

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}$$

for $k_1, \ldots, k_r \in \mathbb{Z}$ (See [1] and [3]. Another generalization is in [2]). Note that when $r = 1$, these definitions reduce to those of the poly-Bernoulli numbers. Similar to $\xi(k; s)$, one can define the function $\xi(k_1, \ldots, k_r; s)$ by

$$\xi(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t - 1} dt \quad (\Re(s) > 1-r)$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$. In a way parallel to deriving (5), a similar result for $\xi(k_1, \ldots, k_r; s)$, which generalizes (3), holds ([6, Remark 2.4]).

About the values at positive integers, Kaneko and Tsumura proved formulas for both $\xi(k_1, \ldots, k_r; s)$ and $\eta(k_1, \ldots, k_r; s)$ ([6, Theorem 2.5]). These formulas have remarkable similarity in that one obtains the formula for $\eta(k_1, \ldots, k_r; s)$ just by replacing multiple zeta values in the one for $\xi(k_1, \ldots, k_r; s)$ with multiple zeta star values. Therefore, the ξ-function (*Arakawa–Kaneko zeta function*) and the η-function (*Kaneko–Tsumura zeta function*) can be regarded as a pair.

Kaneko and Tsumura also considered the function $\eta(-k_1, \ldots, -k_r; s)$ for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, which is defined by (4) with $k_1, \ldots, k_r \in \mathbb{Z}_{\leq 0}$. They established the analytic continuation of $\eta(-k_1, \ldots, -k_r; s)$ to an entire function, and showed that

$$\eta(-k_1, \ldots, -k_r; -n) = B_n^{(-k_1, \ldots, -k_r)} \quad (n \in \mathbb{Z}_{\geq 0})$$

([6, Theorem 4.4]). On the other hand, they defined the function $\tilde{\xi}(-k_1, \ldots, -k_r; s)$ by

$$\tilde{\xi}(-k_1, \ldots, -k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{-k_1, \ldots, -k_r}(1 - e^t)}{e^{-t} - 1} dt \quad (\Re(s) > 1 - r)$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1 \ldots, k_r) \neq (0, \ldots, 0)$. It is proved that $\tilde{\xi}(-k_1, \ldots, -k_r; s)$ is also analytically continued to an entire function, and satisfies

$$\tilde{\xi}(-k_1, \ldots, -k_r; -n) = C_n^{(-k_1, \ldots, -k_r)} \quad (n \in \mathbb{Z}_{\geq 0}).$$

Further Kaneko and Tsumura considered a multi-variable case. Specifically, they introduced more general Bernoulli numbers $\{B_{n_1, \ldots, n_r}^{(s_1, \ldots, s_r), (d)}\}$ called the *multi-indexed poly-Bernoulli numbers*, and defined the function $\eta(-k_1, \ldots, -k_r; s_1, \ldots, s_r)$ whose values at non-positive integers are written by these Bernoulli numbers (We state these definitions in section 2). Then, they obtained the duality formula for the multi-indexed poly-Bernoulli numbers ([6, Theorem 5.4]), which is a generalization of that for the poly-Bernoulli numbers of B-type (1).

On the other hand, the multi-indexed poly-Bernoulli numbers *of C-type* $\{C_{n_1, \ldots, n_r}^{(s_1, \ldots, s_r), (d)}\}$ and the *multi-variable Arakawa–Kaneko zeta function for non-positive indices* $\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r)$ are studied for the first time in this paper. Specifically, we construct these objects, obtain the analytic continuation of the function $\tilde{\xi}$, and establish a certain duality formula for the numbers $\{C_{n_1, \ldots, n_r}^{(s_1, \ldots, s_r), (d)}\}$, which is a generalization of that for the poly-Bernoulli numbers of C-type (2).

This paper is organized as follows. In section 2, recalling the research of Kaneko and Tsumura [6], we define the multi-indexed poly-Bernoulli numbers of C-type and the multi-variable Arakawa–Kaneko zeta function for non-positive indices. In section 3, we state a key lemma, and give the analytic continuation of $\tilde{\xi}$. In section 4, we derive a certain duality formula for the multi-indexed poly-Bernoulli numbers of C-type with a related formula.

2. Definitions of the multi-indexed poly-Bernoulli numbers of C-type and the multi-variable Arakawa–Kaneko zeta function for non-positive indices

We first recall two types of the multi-variable multiple polylogarithm functions defined by

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) := \sum_{0<m_1<\cdots<m_r} \frac{z_1^{m_1} z_2^{m_2}\cdots z_r^{m_r}}{m_1^{s_1}\cdots m_r^{s_r}}, \tag{6}$$

$$\begin{aligned}\mathrm{Li}^{\sqcup\!\sqcup}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) &:= \sum_{0<m_1<\cdots<m_r} \frac{z_1^{m_1} z_2^{m_2-m_1}\cdots z_r^{m_r-m_{r-1}}}{m_1^{s_1}\cdots m_r^{s_r}} \\ &= \sum_{l_1,\ldots,l_r\geq 1} \frac{z_1^{l_1} z_2^{l_2}\cdots z_r^{l_r}}{l_1^{s_1}(l_1+l_2)^{s_2}\cdots(l_1+\cdots+l_r)^{s_r}}\end{aligned}$$

for $s_1,\ldots,s_r \in \mathbb{C}$ and $z_1,\ldots,z_r \in \mathbb{C}$ with $|z_j| < 1$ $(1 \leq j \leq r)$. By the definition, it holds that

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \mathrm{Li}^{\sqcup\!\sqcup}_{s_1,\ldots,s_r}\left(\prod_{\nu=1}^{r} z_\nu, \prod_{\nu=2}^{r} z_\nu, \ldots, z_r\right).$$

Kaneko and Tsumura introduced the multi-indexed poly-Bernoulli numbers (*of B-type*) $\{B^{(s_1,\ldots,s_r),(d)}_{n_1,\ldots,n_r}\}$ by

$$\frac{\mathrm{Li}^{\sqcup\!\sqcup}_{s_1,\ldots,s_r}(1-e^{-\sum_{\nu=1}^{r} t_\nu}, 1-e^{-\sum_{\nu=2}^{r} t_\nu}, \ldots, 1-e^{-t_r})}{\prod_{j=1}^{d}(1-e^{-\sum_{\nu=j}^{r} t_\nu})} = \sum_{n_1,\ldots,n_r\geq 0} B^{(s_1,\ldots,s_r),(d)}_{n_1,\ldots,n_r} \prod_{j=1}^{r} \frac{t_j^{n_j}}{n_j!}$$

([6, Definition 5.1]). When $n_1 = \cdots = n_{r-1} = 0$ and $d = 1$, these numbers reduce to the multi-poly-Bernoulli numbers of B-type, namely, $B^{(k_1,\ldots,k_r),(1)}_{0,\ldots,0,n} = B^{(k_1,\ldots,k_r)}_{n}$ for $k_1,\ldots,k_r \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Then, for $m_1,\ldots,m_r,k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$, it holds that

$$B^{(-k_1,\ldots,-k_r),(r)}_{m_1,\ldots,m_r} = B^{(-m_1,\ldots,-m_r),(r)}_{k_1,\ldots,k_r} \tag{7}$$

([6, Theorem 5.4]). The equation (7) is a beautiful generalization of the duality formula (1). To show (7), Kaneko and Tsumura considered the function $\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$ defined by

$$\begin{aligned}\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r) &:= \frac{1}{\prod_{j=1}^{r}\Gamma(s_j)} \int_0^\infty\cdots\int_0^\infty \prod_{j=1}^{r} t_j^{s_j-1} \\ &\times \frac{\mathrm{Li}^{\sqcup\!\sqcup}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^{r} t_\nu}, 1-e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1-e^{t_r})}{\prod_{j=1}^{r}(1-e^{\sum_{\nu=j}^{r} t_\nu})} \prod_{j=1}^{r} dt_j\end{aligned} \tag{8}$$

for $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$ and $s_1,\ldots,s_r \in \mathbb{C}$ with $\Re(s_j) > 0$ $(j = 1,\ldots,r)$ ([6, Definition 5.6, Theorem 5.7, 5.10]).

Similarly, we introduce the *multi-indexed poly-Bernoulli numbers of C-type* $\{C^{(s_1,\ldots,s_r),(d)}_{n_1,\ldots,n_r}\}$.

DEFINITION 1. For $s_1, \ldots, s_r \in \mathbb{C}$, define the *multi-indexed poly-Bernoulli numbers of C-type* $\{C_{n_1,\ldots,n_r}^{(s_1,\ldots,s_r),(d)}\}$ by

$$\frac{\mathrm{Li}^{ш}_{s_1,\ldots,s_r}(1-e^{-\sum_{\nu=1}^r t_\nu}, 1-e^{-\sum_{\nu=2}^r t_\nu}, \ldots, 1-e^{-t_r})}{\prod_{j=1}^d (e^{\sum_{\nu=j}^r t_\nu}-1)} = \sum_{n_1,\ldots,n_r \geq 0} C_{n_1,\ldots,n_r}^{(s_1,\ldots,s_r),(d)} \prod_{j=1}^r \frac{t_j^{n_j}}{n_j!}. \tag{9}$$

When $n_1 = \cdots = n_{r-1} = 0$ and $d = 1$, these numbers reduce to the multi-poly-Bernoulli numbers of C-type, namely, $C_{0,\ldots,0,n}^{(k_1,\ldots,k_r),(1)} = C_n^{(k_1,\ldots,k_r)}$ for $k_1, \ldots, k_r \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. To study the numbers $\{C_{n_1,\ldots,n_r}^{(s_1,\ldots,s_r),(d)}\}$, we consider the *multi-variable Arakawa–Kaneko zeta function for non-positive indices* $\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$.

DEFINITION 2. Let $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ with $k_1 \geq 1$ and $d \in \{1, \ldots, r\}$. Then define

$$\begin{aligned}\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d) &:= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r t_j^{s_j-1} \\ &\times \frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu}, 1-e^{\sum_{\nu=2}^r t_\nu}, \ldots, 1-e^{t_r})}{\prod_{j=1}^d (e^{-\sum_{\nu=j}^r t_\nu}-1)} \prod_{j=1}^r dt_j\end{aligned} \tag{10}$$

for $s_1, \ldots, s_r \in \mathbb{C}$ with $\Re(s_j) > 0$ $(j = 1, \ldots, r)$.

The definition (10) in the case $d = r$ corresponds to what we want to regard as $\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r)$.

To guarantee the convergence of the integral (10), we note that the multi-variable multiple polylogarithms with non-positive indices become rational functions of the following form:

LEMMA 1 ([6, Theorem 5.5]). *For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $P(x_1 \ldots, x_r; k_1, \ldots, k_r) \in \mathbb{Z}[x_1, \ldots, x_r]$ such that*

$$\mathrm{Li}^*_{-k_1,\ldots,-k_r}(z_1, \ldots, z_r) = \frac{P(\prod_{\nu=1}^r z_\nu, \prod_{\nu=2}^r z_\nu, \ldots, z_r; k_1, \ldots, k_r)}{\prod_{j=1}^r (1-\prod_{\nu=j}^r z_\nu)^{\sum_{\nu=j}^r k_\nu + 1}}, \tag{11}$$

$$\deg_{x_j} P(x_1, \ldots, x_r; k_1, \ldots, k_r) \leq \sum_{\nu=j}^r k_\nu + 1 \quad (j = 1, \ldots, r), \tag{12}$$

$$x_1 \cdots x_r | P(x_1, \ldots, x_r; k_1, \ldots, k_r). \tag{13}$$

Set $y_j = \prod_{\nu=j}^r z_\nu$ $(j = 1, \ldots, r)$. Then (11) *implies*

$$\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(y_1, \ldots, y_r) = \frac{P(y_1, \ldots, y_r; k_1, \ldots, k_r)}{\prod_{j=1}^r (1-y_j)^{\sum_{\nu=j}^r k_\nu + 1}}.$$

In particular, if $k_1 \geq 1$, we obtain

$$\deg_{x_1} P(x_1, \ldots, x_r; k_1, \ldots, k_r) \leq \sum_{\nu=1}^r k_\nu, \tag{14}$$

which is not mentioned in [6]. We state a proof of Lemma 1 to emphasize this assertion. Firstly, we put $D_i = z_i \frac{\partial}{\partial z_i}$ $(i = 1, \ldots, r)$. Then D_i is a derivation and satisfies

$$D_i\left(\prod_{\nu=j}^{r} z_\nu\right) = \begin{cases} \prod\limits_{\nu=j}^{r} z_\nu & (i \in \{j, \ldots, r\}) \\ 0 & (\text{otherwise}) \end{cases} .$$

By the definition (6), it holds that

$$\mathrm{Li}^*_{-k_1,\ldots,-k_r}(z_1, \ldots, z_r) = D_r^{k_r}\left(D_{r-1}^{k_{r-1}}\left(\cdots D_1^{k_1}\left(\frac{\prod_{\nu=1}^r z_\nu}{1-\prod_{\nu=1}^r z_\nu}\right)\cdots\frac{z_{r-1}z_r}{1-z_{r-1}z_r}\right)\frac{z_r}{1-z_r}\right) .$$

We note that the equation:

$$D_i\left(\frac{(\prod_{\nu=j}^r z_\nu)^l}{(1-\prod_{\nu=j}^r z_\nu)^k}\right) = \left(l + (k-l)\prod_{\nu=j}^{r} z_\nu\right)\frac{(\prod_{\nu=j}^r z_\nu)^l}{(1-\prod_{\nu=j}^r z_\nu)^{k+1}} \tag{15}$$

holds for $i \in \{j, \ldots, r\}$, $l \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 1}$. Since (15) is proved by a direct calculation, we omit a proof.

PROOF OF LEMMA 1. We use the induction on r.
The case $r = 1$: For $k = 0$, we observe $\mathrm{Li}_0(z) = \frac{z}{1-z}$. Therefore, the assertion is obvious. For $k \geq 1$, we suppose

$$\mathrm{Li}_{-k+1}(z) = \sum_{i=1}^{k} a_i \frac{z^i}{(1-z)^k}$$

for some $a_i \in \mathbb{Z}$. Then by (15), it holds that

$$\begin{aligned}\mathrm{Li}_{-k}(z) &= z\frac{d}{dz}(\mathrm{Li}_{-k+1}(z)) \\ &= \sum_{i=1}^{k} a_i(i + (k-i)z)\frac{z^i}{(1-z)^{k+1}} \\ &= \frac{1}{(1-z)^{k+1}}\left(a_1 z + \sum_{i=2}^{k}(ia_i + (k-i+1)a_{i-1})z^i\right) .\end{aligned}$$

Therefore, setting $P(x;k) := a_1 x + \sum_{i=2}^{k}(ia_i + (k-i+1)a_{i-1})x^i$, we can verify (11), (13) and (14).
The case $r \geq 2$: For $k_r = 0$, we observe

$$\begin{aligned}\mathrm{Li}^*_{-k_1,\ldots,-k_{r-1},0}(z_1, \ldots, z_r) &= \frac{z_r}{1-z_r}\mathrm{Li}^*_{-k_1,\ldots,-k_{r-2},-k_{r-1}}(z_1, \ldots, z_{r-2}, z_{r-1}z_r) \\ &= \frac{z_r}{1-z_r}\frac{P(\prod_{\nu=1}^r z_\nu, \prod_{\nu=2}^r z_\nu, \ldots, z_{r-1}z_r; k_1, \ldots, k_{r-1})}{\prod_{j=1}^{r-1}(1-\prod_{\nu=j}^r z_\nu)^{\sum_{\nu=j}^{r-1} k_\nu + 1}} ,\end{aligned}$$

where

$$P(x_1,\dots,x_{r-1};k_1,\dots,k_{r-1})=\sum_{i_1=1}^{k_1+\cdots+k_{r-1}+1}\sum_{i_2=1}^{k_2+\cdots+k_{r-1}+1}\cdots\sum_{i_{r-1}=1}^{k_{r-1}+1}a(i_1,\dots,i_{r-1})\prod_{j=1}^{r-1}x_j^{i_j}$$

for some $a(i_1,\dots,i_{r-1})\in\mathbb{Z}$. We set

$$P(x_1,\dots,x_r;k_1,\dots,k_{r-1},0):=x_rP(x_1,\dots,x_{r-1};k_1,\dots,k_{r-1}). \tag{16}$$

Then (16) satisfies (11)–(13). The coefficients of $x_1^{k_1+\cdots+k_r+1}\prod_{j=2}^r x_j^{i_j}$ in (16) are multiples of $a(k_1+\cdots+k_{r-1}+1,i_2,\dots,i_{r-1})$. When $k_1\geq 1$, they are zeros by the induction hypothesis. Hence (14) holds. For $k_r\geq 1$, we suppose

$$\begin{aligned}&\mathrm{Li}^*_{-k_1,\dots,-k_r+1}(z_1,\dots,z_r)\\&\quad=\sum_{i_1=1}^{k_1+\cdots+k_r}\sum_{i_2=1}^{k_2+\cdots+k_r}\cdots\sum_{i_r=1}^{k_r}a(i_1,\dots,i_r)\prod_{j=1}^r\frac{(\prod_{\nu=j}^r z_\nu)^{i_j}}{(1-\prod_{\nu=j}^r z_\nu)^{k_j+\cdots+k_r}}\end{aligned}$$

for some $a(i_1,\dots,i_r)\in\mathbb{Z}$. Applying (15), it holds that

$$\begin{aligned}&\mathrm{Li}^*_{-k_1,\dots,-k_r}(z_1,\dots,z_r)\\&=D_r\left(\mathrm{Li}^*_{-k_1,\dots,-k_r+1}(z_1,\dots,z_r)\right)\\&=\sum_{i_1=1}^{k_1+\cdots+k_r}\sum_{i_2=1}^{k_2+\cdots+k_r}\cdots\sum_{i_r=1}^{k_r}a(i_1,\dots,i_r)\\&\quad\times\sum_{p=1}^r(i_p-(k_p+\cdots+k_r-i_p)\prod_{\nu=p}^r z_\nu)\\&\quad\times\frac{(\prod_{\nu=p}^r z_\nu)^{i_p}}{(1-\prod_{\nu=p}^r z_\nu)^{k_p+\cdots+k_r+1}}\prod_{\substack{j=1\\j\neq p}}^r\frac{(\prod_{\nu=j}^r z_\nu)^{i_j}}{(1-\prod_{\nu=j}^r z_\nu)^{k_j+\cdots+k_r}}\\&=\frac{1}{\prod_{j=1}^r(1-\prod_{\nu=j}^r z_\nu)^{k_j+\cdots+k_r+1}}\sum_{i_1=1}^{k_1+\cdots+k_r}\sum_{i_2=1}^{k_2+\cdots+k_r}\cdots\sum_{i_r=1}^{k_r}a(i_1,\dots,i_r)\\&\quad\times\sum_{p=1}^r\left(i_p-(k_p+\cdots+k_r-i_p)\prod_{\nu=p}^r z_\nu\right)\prod_{\substack{j=1\\j\neq p}}^r\left(1-\prod_{\nu=j}^r z_\nu\right)\prod_{j=1}^r\left(\prod_{\nu=j}^r z_\nu\right)^{i_j}.\end{aligned}$$

Therefore, setting

$$\begin{aligned}&P(x_1,\dots,x_r;k_1,\dots,k_r)\\&:=\sum_{i_1=1}^{k_1+\cdots+k_r}\sum_{i_2=1}^{k_2+\cdots+k_r}\cdots\sum_{i_r=1}^{k_r}a(i_1,\dots,i_r)\end{aligned}$$

$$\times \sum_{p=1}^{r} \bigl(i_p - (k_p + \cdots + k_r - i_p)x_p\bigr) \prod_{\substack{j=1 \\ j\neq p}}^{r} \bigl(1 - x_j\bigr) \prod_{j=1}^{r} x_j^{i_j}\,, \tag{17}$$

we can verify (11)–(13). The coefficients of $x_1^{k_1+\cdots+k_r+1}\prod_{j=2}^{r} x_j^{i_j}$ in (17) are multiples of $a(k_1 + \cdots + k_r, i_2, \ldots, i_r)$. When $k_1 \geq 1$, they are zeros by the induction hypothesis. Hence (14) holds. This ends the proof. $\square$

By Lemma 1, the integrand of (10) is evaluated as

$$\prod_{j=1}^{r} t_j^{s_j-1} \frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)}$$

$$= \prod_{j=1}^{r} t_j^{s_j-1} \frac{P(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{r} e^{\sum_{\nu=j}^{r} t_\nu(\sum_{\nu=j}^{r} k_\nu+1)}} \frac{1}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)}\,,$$

where

$$P(x_1, \ldots, x_r; k_1, \ldots, k_r) = \sum_{i_1=1}^{k_1+\cdots+k_r+1} \sum_{i_2=1}^{k_2+\cdots+k_r+1} \cdots \sum_{i_r=1}^{k_r+1} a(i_1, \ldots, i_r) \prod_{j=1}^{r} x_j^{i_j}$$

for some $a(i_1, \ldots, i_r) \in \mathbb{Z}$. Then

$$\times \prod_{j=1}^{r} t_j^{s_j-1} \frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)}$$

$$= \prod_{j=1}^{r} t_j^{s_j-1} \sum_{i_1=1}^{k_1+\cdots+k_r+1} \sum_{i_2=1}^{k_2+\cdots+k_r+1} \cdots \sum_{i_r=1}^{k_r+1} a(i_1, \ldots, i_r)$$

$$\times \prod_{j=1}^{d} (e^{-\sum_{\nu=j}^{r} t_\nu} - 1)^{i_j-1} e^{-\sum_{\nu=j}^{r} t_\nu(\sum_{\nu=j}^{r} k_\nu - i_j + 1)}$$

$$\times \prod_{j=d+1}^{r} (e^{-\sum_{\nu=j}^{r} t_\nu} - 1)^{i_j} e^{-\sum_{\nu=j}^{r} t_\nu(\sum_{\nu=j}^{r} k_\nu - i_j + 1)}$$

$$\in \prod_{j=1}^{r} t_j^{s_j-1} e^{-t_j} \cdot \mathbb{Z}[e^{-t_1}, \ldots, e^{-t_r}]\,, \tag{18}$$

since $a(k_1 + \cdots + k_r + 1, i_2, \ldots, i_r) = 0$ for all $i_2, \ldots, i_r$. Therefore, the integral (10) is bounded by a finite sum of products of $\int_0^\infty t^{s-1}e^{-t}dt = \Gamma(s)$, which is convergent for $\Re(s) > 0$.

3. The analytic continuation of the function $\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$

Our first main theorem is stated as follows:

THEOREM 1. *Let $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ with $k_1 \geq 1$ and $d \in \{1, \ldots, r\}$. Then, $\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$ can be analytically continued to an entire function on the whole complex space, and it holds that*

$$\tilde{\xi}(-k_1, \ldots, -k_r; -m_1, \ldots, -m_r; d) = C^{(-k_1,\ldots,-k_r),(d)}_{m_1,\ldots,m_r} \quad (m_1, \ldots, m_r \in \mathbb{Z}_{\geq 0}).$$

Proof. Let

$$H(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$$
$$:= \int_{\mathcal{C}^r} \prod_{j=1}^{r} t_j^{s_j-1} \frac{\mathrm{Li}^{\sqcup\!\sqcup}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)} \prod_{j=1}^{r} dt_j, \tag{19}$$

where $\mathcal{C}$ is the path consisting of the positive real axis from the infinity to sufficiently small ε, a counter clockwise circle C_ε around the origin of radius ε, and the positive real axis from ε to the infinity. Since the evaluation (18) holds and the integrand on the right hand side of (19) has no singularity on $\mathcal{C}^r$, the function $H(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$ is entire. Transform $H(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$ as

$$H(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d) = \prod_{j=1}^{r}(e^{2\pi\sqrt{-1}s_j} - 1)$$
$$\times \int_{\varepsilon}^{\infty} \cdots \int_{\varepsilon}^{\infty} \prod_{j=1}^{r} t_j^{s_j-1} \frac{\mathrm{Li}^{\sqcup\!\sqcup}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)} \prod_{j=1}^{r} dt_j$$
$$+ \int_{(C_\varepsilon)^r} \prod_{j=1}^{r} t_j^{s_j-1} \frac{\mathrm{Li}^{\sqcup\!\sqcup}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, 1 - e^{\sum_{\nu=2}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{d}(e^{-\sum_{\nu=j}^{r} t_\nu} - 1)} \prod_{j=1}^{r} dt_j$$
$$+ (\texttt{ the other terms }), \tag{20}$$

where `the other terms` are integrals whose paths consist of the intervals (ε, ∞) and the circles C_ε. Suppose $\Re(s_j) > 0$ $(j = 1, \ldots, r)$, then each term except for the first term on the right hand side of (20) tends to zero as $\varepsilon \to 0$. Hence it holds that

$$\tilde{\xi}(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d)$$
$$= \frac{1}{\prod_{j=1}^{r}(e^{2\pi\sqrt{-1}s_j} - 1)\Gamma(s_j)} H(-k_1, \ldots, -k_r; s_1, \ldots, s_r; d) \tag{21}$$

which can be analytically continued to $\mathbb{C}^r$, and is entire. Set $s_j = -m_j \in \mathbb{Z}_{\leq 0}$ $(j = 1, \ldots, r)$. Then, by (9), (20) and (21), we obtain

$$\tilde{\xi}(-k_1, \ldots, -k_r; -m_1, \ldots, -m_r; d)$$
$$= \prod_{j=1}^{r} \frac{(-1)^{m_j}}{2\pi\sqrt{-1}} m_j! \int_{(C_\varepsilon)^r} \prod_{j=1}^{r} t_j^{-m_j-1} \sum_{n_1,\ldots,n_r \geq 0} C^{(-k_1,\ldots,-k_r),(d)}_{n_1,\ldots,n_r} \prod_{j=1}^{r} \frac{(-t_j)^{n_j}}{n_j!} \prod_{j=1}^{r} dt_j$$
$$= C^{(-k_1,\ldots,-k_r),(d)}_{m_1,\ldots,m_r}.$$

This completes the proof. □

4. Duality formula for the numbers $\{C^{(s_1,\ldots,s_r),(d)}_{n_1,\ldots,n_r}\}$

In this section, we show a kind of duality formula for the multi-indexed poly-Bernoulli numbers of C-type.

LEMMA 2 (cf. [6, Lemma 5.9]). *It holds that*

$$\sum_{k_1,\ldots,k_r\geq 0}\mathrm{Li}^{⧢}_{-k_1-1,-k_2,\ldots,-k_r}(z_1,\ldots,z_r)\prod_{j=1}^{r}\frac{x_j^{k_j}}{k_j!}=\frac{z_1e^{\sum_{\nu=1}^{r}x_\nu}}{(1-z_1e^{\sum_{\nu=1}^{r}x_\nu})^2}\prod_{j=2}^{r}\frac{z_je^{\sum_{\nu=j}^{r}x_\nu}}{1-z_je^{\sum_{\nu=j}^{r}x_\nu}}.$$

Proof. The calculation:

$$\begin{aligned}\sum_{k_1,\ldots,k_r\geq 0}\mathrm{Li}^{⧢}_{-k_1-1,-k_2,\ldots,-k_r}(z_1,\ldots,z_r)\prod_{j=1}^{r}\frac{x_j^{k_j}}{k_j!}&=\sum_{k_1,\ldots,k_r\geq 0}\sum_{l_1,\ldots,l_r\geq 1}l_1\prod_{j=1}^{r}z_j^{l_j}\frac{((\sum_{\nu=1}^{j}l_\nu)x_j)^{k_j}}{k_j!}\\&=\sum_{l_1,\ldots,l_r\geq 1}l_1\prod_{j=1}^{r}z_j^{l_j}e^{(\sum_{\nu=1}^{j}l_\nu)x_j}\\&=\sum_{l_1,\ldots,l_r\geq 1}l_1\prod_{j=1}^{r}(z_je^{\sum_{\nu=j}^{r}x_\nu})^{l_j}\end{aligned}$$

with the equations:

$$\sum_{l=1}^{\infty}x^l=\frac{x}{1-x},\quad \sum_{l=1}^{\infty}lx^l=\frac{x}{(1-x)^2}$$

gives the desired result. □

Our second main theorem is stated as follows:

THEOREM 2. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$, it holds that*

$$\tilde{\xi}(-k_1-1,-k_2,\ldots,-k_r;s_1,\ldots,s_r;r)=\sum_{a=1}^{r}\sum_{1=i_1<i_2<\cdots<i_a\leq r}C^{(\mathtt{i}(s_1-1,s_2,\ldots,s_r)),(1)}_{\mathtt{i}(k_1,\ldots,k_r)},\tag{22}$$

where

$$\mathtt{i}(s_1,\ldots,s_r):=(s_1+\cdots+s_{i_2-1},s_{i_2}+\cdots+s_{i_3-1},\ldots,s_{i_{a-1}}+\cdots+s_{i_a-1},s_{i_a}+\cdots+s_r).$$

Therefore, for $m_1,\ldots,m_r\in\mathbb{Z}_{\geq 0}$, it holds that

$$C^{(-k_1-1,-k_2,\ldots,-k_r),(r)}_{m_1,\ldots,m_r}=\sum_{a=1}^{r}\sum_{1=i_1<i_2<\cdots<i_a\leq r}C^{(\mathtt{i}(-m_1-1,-m_2,\ldots,-m_r)),(1)}_{\mathtt{i}(k_1,\ldots,k_r)}.\tag{23}$$

In the case $r=1$, (23) deduces the duality formula for the poly-Bernoulli numbers of C-type (2).

Proof. We may assume that $\Re(s_j)>0$ $(j=1,\ldots,r)$ because of the identity theorem. Let

$$\mathcal{G}(x_1,\ldots,x_r;s_1,\ldots,s_r):=\sum_{k_1,\ldots,k_r\geq 0}\tilde{\xi}(-k_1-1,-k_2,\ldots,-k_r;s_1,\ldots,s_r;r)\prod_{j=1}^{r}\frac{x_j^{k_j}}{k_j!}\,.$$

Applying Lemma 2, we obtain

$$\begin{aligned}
&\mathcal{G}(x_1,\ldots,x_r;s_1,\ldots,s_r)\\
&=\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}t_j^{s_j-1}\frac{1}{\prod_{j=1}^{r}(e^{-\sum_{\nu=j}^{r}t_\nu}-1)}\\
&\quad\times\frac{e^{\sum_{\nu=1}^{r}x_\nu}(1-e^{\sum_{\nu=1}^{r}t_\nu})}{(1-e^{\sum_{\nu=1}^{r}x_\nu}(1-e^{\sum_{\nu=1}^{r}t_\nu}))^2}\prod_{j=2}^{r}\frac{e^{\sum_{\nu=j}^{r}x_\nu}(1-e^{\sum_{\nu=j}^{r}t_\nu})}{1-e^{\sum_{\nu=j}^{r}x_\nu}(1-e^{\sum_{\nu=j}^{r}t_\nu})}\prod_{j=1}^{r}dt_j\\
&=\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}t_j^{s_j-1}e^{-\sum_{\nu=1}^{r}t_\nu}e^{-\sum_{\nu=1}^{r}x_\nu}\\
&\quad\times\frac{1}{(1-e^{-\sum_{\nu=1}^{r}t_\nu}(1-e^{-\sum_{\nu=1}^{r}x_\nu}))^2}\prod_{j=2}^{r}\frac{1}{1-e^{-\sum_{\nu=j}^{r}t_\nu}(1-e^{-\sum_{\nu=j}^{r}x_\nu})}\prod_{j=1}^{r}dt_j\\
&=\frac{1}{e^{\sum_{\nu=1}^{r}x_\nu}-1}\sum_{\substack{m_1\geq 1\\ m_2,\ldots,m_r\geq 0}}m_1\prod_{j=1}^{r}(1-e^{-\sum_{\nu=j}^{r}x_\nu})^{m_j}\\
&\quad\times\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}t_j^{s_j-1}e^{-m_j\sum_{\nu=j}^{r}t_\nu}dt_j\,.
\end{aligned}\tag{24}$$

We see that the integrand on the last line can be rewritten as

$$\prod_{j=1}^{r}t_j^{s_j-1}e^{-t_j\sum_{\nu=1}^{j}m_\nu}\,.$$

Substituting

$$n^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-nt}dt$$

into (24), it holds that

$$\begin{aligned}
&\mathcal{G}(x_1,\ldots,x_r;s_1,\ldots,s_r)\\
&=\frac{1}{e^{\sum_{\nu=1}^{r}x_\nu}-1}\sum_{\substack{m_1\geq 1\\ m_2,\ldots,m_r\geq 0}}\prod_{j=1}^{r}\left(\sum_{\nu=1}^{j}m_\nu\right)^{-s_j}m_1\prod_{j=1}^{r}(1-e^{-\sum_{\nu=j}^{r}x_\nu})^{m_j}\\
&=\frac{1}{e^{\sum_{\nu=1}^{r}x_\nu}-1}\sum_{a=1}^{r}\sum_{1=i_1<i_2<\cdots<i_a\leq r}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{m_1,m_{i_2},\dots,m_{i_a}\ge 1} m_1 \prod_{j=1}^{a}\left(\sum_{\nu=1}^{j} m_{i_\nu}\right)^{-\sum_{\nu=i_j}^{i_{j+1}-1} s_\nu} (1-e^{-\sum_{\nu=i_j}^{r} x_\nu})^{m_{i_j}} \\
&= \sum_{a=1}^{r} \sum_{1=i_1<i_2<\cdots<i_a\le r} \frac{1}{e^{\sum_{\nu=1}^{r} x_\nu}-1} \\
&\quad \times \mathrm{Li}^{\sqcup\!\sqcup}_{\mathtt{i}(s_1-1,s_2,\dots,s_r)}(1-e^{-\sum_{\nu=1}^{r} x_\nu}, 1-e^{-\sum_{\nu=i_2}^{r} x_\nu}, \dots, 1-e^{-\sum_{\nu=i_a}^{r} x_\nu}) \\
&= \sum_{a=1}^{r} \sum_{1=i_1<i_2<\cdots<i_a\le r} \sum_{n_1,\dots,n_a\ge 0} C_{n_1,\dots,n_a}^{(\mathtt{i}(s_1-1,s_2,\dots,s_r)),(1)} \prod_{j=1}^{a} \frac{(\sum_{\nu=i_j}^{i_{j+1}-1} x_\nu)^{n_j}}{n_j!} \\
&= \sum_{m_1,\dots,m_r\ge 0} \sum_{a=1}^{r} \sum_{1=i_1<i_2<\cdots<i_a\le r} C_{\mathtt{i}(m_1,\dots,m_r)}^{\mathtt{i}(s_1-1,s_2,\dots,s_r),(1)} \prod_{j=1}^{r} \frac{x_j^{m_j}}{m_j!}\,,
\end{aligned}
$$

where we regard $i_{a+1}-1$ as r. Therefore, we obtain (22). □

EXAMPLE 1. We can easily see that

$$
\begin{aligned}
\mathrm{Li}^{\sqcup\!\sqcup}_{-2,0}(z_1,z_2) &= \frac{z_1z_2(1+z_1)}{(1-z_1)^3(1-z_2)}\,, \\
\mathrm{Li}^{\sqcup\!\sqcup}_{-1,-1}(z_1,z_2) &= \frac{z_1z_2(2-z_1z_2-z_2)}{(1-z_1)^3(1-z_2)^2}\,, \\
\mathrm{Li}_{-2}(z) &= \frac{z(1+z)}{(1-z)^3}\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&C_{m,n}^{(-2,0),(2)} = 2^{m+n+1}-1\,, \\
&C_{m,n}^{(-1,-1),(1)} = 2^{m+1}3^n + 3^n - 2^{m+n+1} - 2^{n+1} + 1\,, \\
&C_n^{(-2)} = 2^{n+1}-1 \ (m,n\in\mathbb{Z}_{\ge 0})
\end{aligned}
$$

and $C_{0,1}^{(-2,0),(2)} = C_{1,0}^{(-1,-1),(1)} + C_1^{(-2)} = 3$ hold. Similarly we obtain, for example,

$$
C_{1,0}^{(-2,-2),(2)} = C_{1,2}^{(-2,0),(1)} + C_3^{(-2)} = 32,\ C_{2,1}^{(-2,-1),(2)} = C_{1,1}^{(-3,-1),(1)} + C_2^{(-4)} = 137\,.
$$

Considering the function $\eta(-k_1,\dots,-k_r;s)$, Kaneko and Tsumura obtained

$$
\eta(-k_1,\dots,-k_r;s) = \mathfrak{B}_{k_1,\dots,k_r}^{(s)}
$$

for $k_1,\dots,k_r\in\mathbb{Z}_{\ge 0}$ and $s\in\mathbb{C}$. Therefore, for $m\in\mathbb{Z}_{\ge 0}$, it holds that

$$
B_m^{(-k_1,\dots,-k_r)} = \mathfrak{B}_{k_1,\dots,k_r}^{(-m)} \tag{25}
$$

([6, Theorem 4.7]). Here, $\{\mathfrak{B}_{n_1,\dots,n_r}^{(s)}\}$ is another generalization of the poly-Bernoulli numbers defined by

$$
\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{l_1,\dots,l_r\ge 1}\frac{\prod_{j=1}^{r}(1-e^{-\sum_{\nu=j}^{r} x_\nu})^{l_j-1}}{(\sum_{\nu=1}^{r} l_\nu - a)^s} = \sum_{n_1,\dots,n_r\ge 0}\mathfrak{B}_{n_1,\dots,n_r}^{(s)}\prod_{j=1}^{r}\frac{x_j^{n_j}}{n_j!}
$$

for $s \in \mathbb{C}$. In the case $r = 1$, we see that $\mathfrak{B}_n^{(k)} = B_n^{(k)}$ for $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Therefore, (25) is viewed as a duality formula which generalizes the duality formula for the poly-Bernoulli numbers of B-type (1). Similarly we define another type of the multi-poly-Bernoulli numbers $\{\mathfrak{C}_{n_1,\ldots,n_r}^{(s)}\}$ by

$$\begin{aligned} &e^{-\sum_{\nu=1}^r x_\nu} \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{l_1,\ldots,l_r \geq 1} l_1 \frac{\prod_{j=1}^r (1 - e^{-\sum_{\nu=j}^r x_\nu})^{l_j - 1}}{(\sum_{\nu=1}^r l_\nu - a)^s} \\ &= \sum_{n_1,\ldots,n_r \geq 0} \mathfrak{C}_{n_1,\ldots,n_r}^{(s)} \prod_{j=1}^r \frac{x_j^{n_j}}{n_j!} \end{aligned} \tag{26}$$

for $s \in \mathbb{C}$. In the case $r = 1$, we see that $\mathfrak{C}_n^{(k)} = C_n^{(k-1)}$ for $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Then we obtain another sort of the duality formula:

THEOREM 3. *For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, it holds that*

$$\tilde{\xi}(-k_1 - 1, -k_2, \ldots, -k_r; s) = \mathfrak{C}_{k_1,\ldots,k_r}^{(s)} . \tag{27}$$

Therefore, for $m \in \mathbb{Z}_{\geq 0}$, it holds that

$$C_m^{(-k_1-1,-k_2,\ldots,-k_r)} = \mathfrak{C}_{k_1,\ldots,k_r}^{(-m)} . \tag{28}$$

In the case $r = 1$, (28) deduces the duality formula for the poly-Bernoulli numbers of C-type (2).

Proof. Let

$$\mathcal{G}(x_1, \ldots, x_r; s) := \sum_{k_1,\ldots,k_r \geq 0} \tilde{\xi}(-k_1 - 1, -k_2, \ldots, -k_r; s) \prod_{j=1}^r \frac{x_j^{k_j}}{k_j!} .$$

Then, we obtain

$$\begin{aligned} &\mathcal{G}(x_1, \ldots, x_r; s) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{e^{-t} - 1} \frac{e^{\sum_{\nu=1}^r x_\nu}(1 - e^t)}{(1 - e^{\sum_{\nu=1}^r x_\nu}(1 - e^t))^2} \prod_{j=2}^r \frac{e^{\sum_{\nu=j}^r x_\nu}(1 - e^t)}{1 - e^{\sum_{\nu=j}^r x_\nu}(1 - e^t)} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (1 - e^t)^{r-1} e^{-rt} e^{-\sum_{\nu=1}^r x_\nu} \\ &\quad \times \frac{1}{(1 - e^{-t}(1 - e^{-\sum_{\nu=1}^r x_\nu}))^2} \prod_{j=2}^r \frac{1}{1 - e^{-t}(1 - e^{-\sum_{\nu=j}^r x_\nu})} dt \\ &= \frac{1}{e^{\sum_{\nu=1}^r x_\nu} - 1} \sum_{\substack{m_1 \geq 1 \\ m_2,\ldots,m_r \geq 0}} m_1 \prod_{j=1}^r (1 - e^{-\sum_{\nu=j}^r x_\nu})^{m_j} \frac{1}{\Gamma(s)} \\ &\quad \times \int_0^\infty t^{s-1} (1 - e^t)^{r-1} e^{-(\sum_{\nu=1}^r m_\nu + r - 1)t} dt \end{aligned}$$

$$= \frac{1}{e^{\sum_{\nu=1}^{r} x_\nu} - 1} \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{\substack{m_1 \geq 1 \\ m_2,\dots,m_r \geq 0}} m_1 \frac{\prod_{j=1}^{r} (1 - e^{-\sum_{\nu=j}^{r} x_\nu})^{m_j}}{(\sum_{\nu=1}^{r} m_\nu + r - 1 - a)^s}$$

$$= e^{-\sum_{\nu=1}^{r} x_\nu} \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{l_1,\dots,l_r \geq 1} l_1 \frac{\prod_{j=1}^{r} (1 - e^{-\sum_{\nu=j}^{r} x_\nu})^{l_j - 1}}{(\sum_{\nu=1}^{r} l_\nu - a)^s},$$

whence (27) follows. □

EXAMPLE 2. When $r = 2$, we can calculate directly from (26) that $\mathfrak{C}_{1,0}^{(s)} = 2 \cdot 3^{-s} - 3 \cdot 2^{-s} + 1$. On the other hand, as $\mathrm{Li}_{-2,0}(z) = z^2(1+z)/(1-z)^4$, we obtain

$$\sum_{m=0}^{\infty} C_m^{(-2,0)} \frac{t^m}{m!} = \frac{\mathrm{Li}_{-2,0}(1 - e^{-t})}{e^t - 1} = \frac{(1 - e^{-t})(2 - e^{-t})}{e^{-3t}} = 2e^{3t} - 3e^{2t} + e^t,$$

whence $C_m^{(-2,0)} = 2 \cdot 3^m - 3 \cdot 2^m + 1$. Thus it holds that $C_m^{(-2,0)} = \mathfrak{C}_{1,0}^{(-m)}$.

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